Lecture 13

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Facility Location & Valid Utility games

Recall:

For cost-minimization games,
$\forall s \in \mathcal{S}, \quad C(s) = \sum_i c_i(s)$
(utilitarian) social cost

$$PoA(\Gamma) = \frac{\max \{cost(s) : s \text{ is an eq.}\}}{\min \{cost(s) : s \in \mathcal{S}\}} \geq 1$$

For utility-maximization games,
$\forall s \in \mathcal{S}, \quad U(s) = \sum_i u_i(s)$
(utilitarian) social utility / welfare

$$PoA(\Gamma) = \frac{\max \{U(s) : s \in \mathcal{S}\}}{\min \{U(s) : s \text{ is an eq.}\}} \geq 1$$

· For a class of games C,
$$PoA(C) = \max_{\Gamma \in C} PoA(\Gamma)$$

Similarly, PoS is the cost/utility of the best equilibrium.

Last time:

(I) For congestion games, $PoA \leq 5/2$
(If s is equilibrium, s^* is min cost profile,
$$cost(s) \leq \sum_i c_i(s_i^*, s_{-i}) \leq \frac{5}{3} cost(s^*) + \frac{1}{3} cost(s)$$
↑ s is equilibrium; ↑ linear utilities, $x(y+1) \leq \frac{5}{3}x^2 + \frac{1}{3}y^2$

$\Rightarrow cost(s) \leq 5/2 \; cost(s^*)$

(II) If $\phi(s)$ is potential fn. for a game,
& $\alpha \phi(s) \leq cost(s) \leq \beta \phi(s)$,
then $PoS \leq \beta / \alpha$
(hence for GCGs, $PoS \leq H(n)$)

Facility Location Games

Consists of:
- k service providers (denoted by set SP)
- n locations (set $L = \mathcal{S}_i \; \forall i$)
- m clients (set C)

$\forall j \in C$, $\pi(j) \in \mathbb{R}_+$ is value client j gets by being served

$\forall l \in C, j \in C$, $c(l, j)$ is cost if client j is served at location j.

Assume: $\forall j, \pi(j) \geq c(l, j) \; \forall l$
For a fixed j, $c(l, j) \neq c(l', j)$

Given $s \in \mathcal{S}$, each SP i has a price $p(i, j)$ for serving client j ($s_i \in L$ is location chosen by SP$_i$).
Let $\min(j, P) = \{i : p(i, j) \text{ is minimum}\}$
Then we say client j chooses SP(i)
$SP(j) = i$ if (I) $i \in \min(j, P)$, and
(II) $c(s_i, j)$ is minimum among all SPs in $\min(j, P)$

for $i \in SP$, $u_i(s) = \sum_{j : SP(j) = i} \left(p(i, j) - c(s_i, j)\right)$

for $j \in C$, $v_j(s) = \pi_j - p(SP(j), j)$
Thus, the social welfare
$$V(s) = \sum_i u_i(s) + \sum_j v_j(s) = \sum_j \left(v_j - c(s_{SP(j)}, j)\right)$$

(this considers the clients as players as well, but with simple strategies, choosing the least-price service provider)

Choosing prices:

Note that technically, prices too are chosen by SPs
(since $u_i(s) = \sum_{j : SP(j) = i} p(i, j)$)

However we will show that given s, i.e., locations for each player, the prices are fixed at equilibrium.

Note: $\forall i, j, \; p_{ij} \geq c(s_i, j)$

(do example)

Then $p(i, j) = \max \{c(s_i, j), \min_{i' \neq i} c(s_{i'}, j)\}$

Claim: If $SP(j) = i$ then (I) $c(s_i, j) = \min_{i'} c(s_{i'}, j)$
(II) $p(i, j) = \min_{i' \neq i} c(s_{i'}, j)$

Proof: (I) Say $SP(j) = i$, but $c(s_i, j) > \min_{i'} c(s_{i'}, j) = c(s_{\hat{i}}, j)$

Then $p(\hat{i}, j) = \max \{c(s_{\hat{i}}, j), \min_{i' \neq \hat{i}} c(s_{i'}, j)\}$
where $c(s_{\hat{i}}, j) < c(s_i, j)$ and $\min_{i' \neq \hat{i}} c(s_{i'}, j) \leq c(s_i, j)$

$\leq c(s_i, j) = p(i, j)$
But then $SP(j) = \hat{i}$, contradiction

(II) easy.

Claim: $V(s)$ is a potential fn. for the facility location game.

For analysis, define location ϕ, $c(\phi, j) = \pi(j)$
Thus if $s_i = \phi$, $p(i, j) = c(\phi, j) = \pi(j)$

Proof: Will first show that
$$C_i(s) - C_i(s_i = \phi, s_{-i}) = V(s) - V(s_i = \phi, s_{-i})$$
$$LHS = \sum_{SP(j) = i} \left(p(i, j) - c(s_i, j)\right)$$
where $p(i, j) = \min_{i' \neq i} c(s_{i'}, j)$ and $c(s_i, j) = \min_{i'} c(s_{i'}, j)$

$$V(s) = \sum_j \left(\pi(j) - \min_{i'} c(s_{i'}, j)\right)$$

$$V(s_i = \phi, s_{-i}) = \sum_j \left(\pi_j - \min_{i' \neq i} c(s_{i'}, j)\right)$$

Thus, $V(s) - V(s_i = \phi, s_{-i}) = \sum_j \left(\min_{i' \neq i} c(s_{i'}, j) - \min_{i'} c(s_{i'}, j)\right)$

For $SP(j) \neq i$, then $\min_{i' \neq i} c(s_{i'}, j) = \min_{i'} c(s_{i'}, j)$

Thus $V(s) - V(s_i = \phi, s_{-i}) = \sum_{j : SP(j) = i} \left(p_{i,j} - c(s_i, j)\right)$
$= C_i(s) - C_i(s_i = \phi, s_{-i})$

The proof of the claim follows. ∎

Lemma: The facility location game has $PoS = 1$
(easy, from above claim)

What about the PoA? For this, we introduce a generalization, called "Valid Utility" games

Valid Utility Games:

Let Γ be a symmetric game, i.e., $\mathcal{S}_i = \mathcal{S}_j \; \forall i, j \in N$.
Let $A := \mathcal{S}_i$.
Then Γ is a VU game if the social welfare fn. $V(s)$ satisfies:

(I) $V : 2^A \to \mathbb{R}$ (i.e., welfare depends only on set of actions taken, not on which player chooses which action)

(II) V is submodular: $\forall T' \subseteq T \subseteq A$, $t \in A$, $t \notin T$
$$V(T \cup \{t\}) - V(T) \leq V(T' \cup \{t\}) - V(T')$$

(III) $u_i(s) \geq V(s) - V(s_i = \phi, s_{-i})$
(utility of a player is at least its marginal contribution to the welfare)

A VU game is <u>basic</u> if (III) is tight, & is <u>monotone</u> if
$V(T') \leq V(T)$ if $T' \subseteq T$.

Claim: The facility location game is a monotone basic VU game
(prove yourself)

Claim: For any monotone VU game, the $PoA \leq 2$

Proof: For a game Γ, let o be an optimal profile, s be an equilibrium
Define $O = \bigcup_i o_i$, $S = \bigcup_i s_i$, & $O^i = \{o_1, o_2, \ldots, o_i\}$
Thus $O^0 = \phi$, $O^k = O$.

Then
$$V(O) - V(S) \leq V(O \cup S) - V(S) \leq V(O^k \cup S) - V(O^{k-1} \cup S) + V(O^{k-1} \cup S) - V(O^{k-2} \cup S) + \ldots + V(O^1 \cup S) - V(O^0 \cup S)$$
(first inequality: ↑ monotonicity)

Now $\forall i \in [k]$,
$$V(O^i \cup S) - V(O^{i-1} \cup S) = V(O^{i-1} \cup S \cup \{o_i\}) - V(O^{i-1} \cup S)$$
$$\leq V(S \cup \{o_i\} \setminus \{s_i\}) - V(S \setminus \{s_i\}) \quad \text{(by SM)}$$
$$\leq u_i(o_i, s_{-i})$$
$$\leq u_i(s)$$

Thus, $V(O) - V(S) \leq u_k(s) + \ldots + u_1(s)$
$= V(S)$
$\Rightarrow V(O) \leq 2 V(S)$ for the PoA ∎